

THE MATRIX EQUATION $XA - AX = f(X)$ WHEN A IS DIAGONALIZABLE

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ABSTRACT.

1. INTRODUCTION

Let K be an algebraically closed field with characteristic 0, $A \in M_n(K)$ and $f \in K[x]$. In [1], we study the matrix equation in the unknown $X \in M_n(K)$

$$(1) \quad XA - AX = f(X).$$

We show that necessarily A and any solution X are simultaneously triangularizable. Yet, we study essentially the solutions that have a sole eigenvalue. Here $A \in M_n(K)$ is a diagonalizable matrix and $f \in K[x]$ is such that its roots in K are known. In the first part, we consider all the solutions of Eq (1) when A has two distinct eigenvalues. Clearly, Eq (1) admit trivial solutions, that is to say, solutions that satisfy $AX = XA$ and $f(X) = 0$. The following result gives a condition on A and f so that there exist non-trivial solutions.

Proposition 1. *Let $A = \text{diag}(\mu I_p, \lambda I_q)$ where λ, μ are distinct elements of K and $\mathcal{T} = f'(f^{-1}(0)) \setminus \{0\}$.*

- i) If $\pm(\lambda - \mu) \notin \mathcal{T}$ then any solution X of Eq (1) satisfies $XA = AX$.*
- ii) If $\mu - \lambda \notin \mathcal{T}$, then the solutions of Eq (1) are in the form*

$$X = \begin{pmatrix} P & Q \\ 0_{q,p} & S \end{pmatrix} \text{ where } f(P) = 0_p, f(Q) = 0_q$$

and there exist such solutions that do not commute with A if and only if

$$\lambda - \mu \in \mathcal{T}.$$

In the second part, we study a model of Eq (1) that admits non-trivial solutions in the sense of Proposition 1 *ii*). For the sake of simplicity, we choose the equation

$$(2) \quad XA - AX = X^2 - X^3 \text{ in the unknown } X \in M_n(K).$$

where $A \in M_n(K)$ is a *diagonalizable* matrix over K . Yet, the results obtained below appear to be generalizable. We need the following

Notation. We can write the complete spectrum $\sigma(A)$ of A in the form

$$\sigma(A) = \bigcup_{r=1}^k B_r \text{ where the } (B_r)_r \text{ are lists that satisfy the following:}$$

- i) For every r , there exists $\lambda_r \in K$, $c_r \in \mathbb{N}$ such that $B_r = \{L_{r,c_r}, \dots, L_{r,1}, L_{r,0}\}$ where, for every $0 \leq i \leq c_r$, the list $L_{r,i}$ is composed of the eigenvalue $\lambda_r + i$ numbered with its multiplicity.*
- ii) If $r \neq s$ and $u \in B_r, v \in B_s$, then $u - v \neq 1$.*

We consider the ordering of the eigenvalues of A induced by such a sequence $(B_r)_r$ and the associated diagonal form of A : there exists an invertible matrix P such

Date: July-12-2014.

2010 Mathematics Subject Classification. Primary 15A15.

that $P^{-1}AP = \bigoplus_{r=1}^k U_r$ where, for every r , $U_r = \text{diag}(B_r)$.

We show that the solutions of Eq (2) admit a decomposition in direct sum.

Theorem 1. *Let A be a diagonalizable matrix and let X be a solution of Eq (2). With the previous notation,*

$$P^{-1}XP = \bigoplus_{r=1}^k X_r \text{ where, for every } r, X_r U_r - U_r X_r = X_r^2 - X_r^3.$$

Moreover, if one adopts the block structure associated with the decomposition of B_r , then X_r is a upper triangular block-matrix, the diagonal of which, being

$$Y_{r,c_r}, \dots, Y_{r,1}, Y_{r,0} \text{ and satisfying: for every } i, Y_{r,i}^2 - Y_{r,i}^3 = 0.$$

Finally, we give the dimension of the algebraic variety of solutions of Eq (2) when A satisfies the condition of Proposition 1 ii).

Theorem 2. *Let $A = \text{diag}(I_p, 0_q)$, where $\frac{1}{5} \leq \frac{p}{q} \leq 5$ and let*

$$\rho = \lfloor (11p^2 + 11q^2 + 2pq)/16 \rfloor.$$

Then the algebraic variety of solutions of Eq (2) has dimension ρ or $\rho - 1$.

2. NON-TRIVIAL SOLUTIONS OF THE MATRIX EQUATION $XA - AX = f(X)$

2.1. Proof of Proposition 1.

Proof. Part 1. • Put $X = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Then Eq (1) can be written

$$(\lambda - \mu) \begin{pmatrix} 0 & Q \\ -R & 0 \end{pmatrix} = f(X).$$

The LHS commute with X , that implies:

$$(3) \quad QR = 0_p, RQ = 0_q, QS = PQ, RP = SR.$$

• We show that $f(X) = \begin{pmatrix} f(P) & f'(P)Q \\ f'(S)R & f(S) \end{pmatrix}$. Indeed, by linearity, it suffices to prove that, for every k , $X^k = \begin{pmatrix} P^k & kP^{k-1}Q \\ kS^{k-1}R & S^k \end{pmatrix}$. If the previous formula is true, then

$$X^{k+1} = \begin{pmatrix} P^{k+1} & P^{k-1}(PQ + kQS) \\ S^{k-1}(kRP + SR) & S^{k+1} \end{pmatrix}$$

and we conclude by recurrence.

• Finally, there are, in addition to relations Eq (3), the following ones

$$(4) \quad f(P) = 0_p, f(S) = 0_q, f'(P)Q = (\lambda - \mu)Q, f'(S)R = (\mu - \lambda)R.$$

We can calculate the solutions in P, S . Note that the sets of eigenvalues $\sigma(P)$ and $\sigma(S)$ are included in $f^{-1}(0)$. We consider such a couple solution. It remains to solve the linear system

$$(f'(P) \otimes I_q)Q = (\lambda - \mu)Q, (f'(S) \otimes I_p)R = (\mu - \lambda)R.$$

The sets $\sigma(f'(P) \otimes I_q)$ and $\sigma(f'(S) \otimes I_p)$ are included in $f'(f^{-1}(0))$. Thus, if $\pm(\lambda - \mu) \notin \mathcal{T}$, then $Q = 0, R = 0$ and $AX = XA$. In the sequel, we assume that $\mu - \lambda \notin \mathcal{T}$; then $R = 0$.

Part 2. One has $f(X) = \begin{pmatrix} f(P) & Z \\ 0 & f(S) \end{pmatrix}$ with

$$f'(\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}, \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$$

where $f'(u, v)$, the Frechet derivative of f in u , is linear in v (see [2, Theorem 4.12]). In particular, Z is a linear function of Q . By identification, we obtain

$$f(P) = 0, f(S) = 0, (\lambda - \mu)Q = Z.$$

We can calculate the solutions in P, S . Note that $\sigma(P)$ and $\sigma(S)$ are included in $f^{-1}(0)$. We consider such a couple solution. It remains to solve a linear equation in the form $\phi(Q) = (\lambda - \mu)Q$ where

$$\sigma(\phi) = (f[\alpha_i, \beta_j])_{i \leq p, j \leq q}, \text{ with } \sigma(P) = (\alpha_i)_{i \leq p} \text{ and } \sigma(S) = (\beta_j)_{j \leq q}$$

(see [2, Theorem 3.9 and the proof of Theorem 3.11]). Here $f[\alpha_i, \beta_j] = 0$ if $\alpha_i \neq \beta_j$ and $f'(\alpha_i)$ else. Since, for every $i \leq p, j \leq q$, $f(\alpha_i) = f(\beta_j) = 0$, we conclude that

$$\sigma(\phi) \subset \mathcal{T} \cup \{0\}.$$

Note that, if $\sigma(P) \cap \sigma(S) = \emptyset$, then $Q = 0$. Finally, there is a non-zero solution in Q if and only if $\lambda - \mu \in \mathcal{T}$; indeed, if there is $\alpha \in K$ such that $\lambda - \mu = f'(\alpha) \neq 0$ and $f(\alpha) = 0$, then we choose $P = \alpha I_p$, $S = \alpha I_q$ and $Q \in \ker(\phi - (\lambda - \mu)I_{pq}) \setminus \{0\}$. \square

Note that the previous results remain valid if we change the polynomial f with a holomorphic function.

Example 1. Let $K = \mathbb{C}$, $A = \text{diag}(0_p, I_q)$, $f(x) = \log(x)$, where \log is the principal logarithm; we seek matrices X that have no eigenvalues on \mathbb{R}^- (the non positive real numbers) such that $XA - AX = \log(X)$. Here $\mathcal{T} = \{1\}$ is equal to $\lambda - \mu$. Thus a solution is in the form $X = \begin{pmatrix} P & Q \\ 0_{q,p} & S \end{pmatrix}$ where $\log(P) = 0_p, \log(S) = 0_q$. Finally, the solutions are in the form

$$X = \begin{pmatrix} I_p & Q \\ 0 & I_q \end{pmatrix} \text{ where } Q \text{ is an arbitrary } p \times q \text{ matrix.}$$

Example 2. Let $K = \mathbb{C}$, $n = 4$, $A = \text{diag}(0_2, I_2)$; we consider the equation $XA - AX = \exp(X) - I_4$. Here $f^{-1}(0) = 2i\pi\mathbb{Z}$ and $\mathcal{T} = \{1\}$ is $\lambda - \mu$ again. Thus a solution is in the form

$$X = \begin{pmatrix} P & Q \\ 0_2 & S \end{pmatrix} \text{ where } \exp(P) = I_2, \exp(S) = I_2, \phi(Q) = Q.$$

Since P, S are diagonalizable, ϕ is diagonalizable too. Up to a change of basis leaving invariant A , we may assume that

$$P = \text{diag}(2i\pi p_1, 2i\pi p_2), S = \text{diag}(2i\pi s_1, 2i\pi s_2).$$

Note that Q depends on $\dim(\ker(\phi) - I)$ free parameters. We consider the following cases:

- If $p_1 = p_2 = s_1 = s_2$, then $\dim(\ker(\phi) - I) = 4$ and Q is an arbitrary matrix.
- If $p_1 = s_1 \neq p_2 = s_2$, then $\dim(\ker(\phi) - I) = 2$ and $Q = \text{diag}(u, v)$ where u, v are arbitrary elements of K .
- If $p_1 = s_1$ is the sole equality, then $\dim(\ker(\phi) - I) = 1$ and $Q = \text{diag}(u, 0)$ where u is arbitrary.

2.2. When f is degenerated function. For special functions f , it can happens that $\{\pm(\lambda - \mu)\} \subset \mathcal{T}$; we shall see that there exist solutions that are not in block-triangular form. We fix P, S such that $f(P) = 0_p, f(S) = 0_q$. Now Q, R satisfy

$$QR = RQ = 0, Q \in \ker(P \otimes I - I \otimes S^T) \cap \ker(f'(P) \otimes I_q - (\lambda - \mu)I_{pq}),$$

$$R \in \ker(S \otimes I - I \otimes P^T) \cap \ker(f'(S) \otimes I_p - (\mu - \lambda)I_{pq}).$$

We are just going to study this typical instance:

let $n = 4$, $A = \text{diag}(2, 2, 0, 0)$ and $f(x) = x^2 - 1$; here $\mathcal{T} = \{2, -2\}$. We choose $P = S = \text{diag}(1, -1)$. Then the associated solutions are the following ones

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & u \\ v & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ where } u, v \text{ are arbitrary elements of } K.$$

In the sequel, we need the following

Lemma 1. *Let X be a solution of the equation*

$$(5) \quad XA - AX = X^p g(X) \text{ where } p \geq 2 \text{ and } g \text{ is a polynomial.}$$

i) *For every polynomial P , $P(X)A - AP(X) = P'(X)X^p g(X)$.*

ii) *For every integer k , $\ker(X^k)$ is A -invariant.*

Proof. i) We check by induction that

$$\text{for every } i \geq 1, \quad X^i A - AX^i = iX^{i-1}X^p g(X).$$

Reasoning by linearity, we deduce the required result.

ii) Let u be such that $X^k u = 0$. Then $X^k A - X^{k-1}AX = X^{p+k-1}g(X)$ and $X^k Au = X^{k-1}AXu$. According to i),

$$X^{k-1}AXu = (AX^{k-1} + (k-1)X^{p+k-2}g(X))Xu = 0.$$

□

3. THE MATRIX EQUATION $XA - AX = X^2 - X^3$

Now on, we study Eq (2) when $f(x) = x^2 - x^3$ and A is diagonalizable; note that 0 is a double root of f and the condition of degeneration obtained in Proposition 1 ii) is here $\lambda - \mu = -1$. Let $u \in \ker(A - \lambda I)$ and X be a solution of Eq (2).

Lemma 2. *Let $\lambda = 0$. Then, for every integer $s \geq 0$, there is a polynomial ϕ_s such that*

$$(A - sI) \cdots (A - I)AXu = \phi_s(X)X^2(I - X)^{s+1}u \text{ and } \phi_s(1) \neq 0.$$

Proof. We show the first condition by recurrence on s . If $s = 0$, then $AXu = X^2(X - I)u$ and $\phi_0(X) = -1$. Assume that the result is true for $s - 1$. Then, according to Lemma 1 ii),

$$(A - sI)\phi_{s-1}(X)X^2(I - X)^s u = \phi_{s-1}(X)X^2(I - X)^s Au + \phi_s(X)X^2(I - X)^{s+1}u$$

$$\text{where } \phi_s(X) = -X^2\phi_{s-1}'(X) - 2X\phi_{s-1}(X) - s(X + 1)\phi_{s-1}(X).$$

It remains to show the second condition. That is equivalent to consider the sequence of polynomials $P_0(x) = 1, P_s(x) = x^2 P_{s-1}'(x) + (sx + 2x + s)P_{s-1}(x)$ and to show that, for every s , $P_s(1) \neq 0$. Clearly, for every s , P_s is a polynomial of degree s such that each of its coefficients is positive. Therefore, by an easy recurrence, we obtain that the sequence $(P_s(1))_s$ is increasing. □

Proposition 2. *One has*

$$Xu \in \bigoplus_{0 \leq i \leq n-1} \ker(A - (i + \lambda)I).$$

Proof. We may assume that $\lambda = 0$.

- Since $X^2 - X^3$ is nilpotent, the eigenvalues of X are 0 or 1. We show that $X^2(I - X)^n = 0$. According to Lemma 1 i), we may assume that $X = \begin{pmatrix} N & 0 \\ 0 & L \end{pmatrix}$ where N and $L - I$ are nilpotent and $A = \begin{pmatrix} D & E \\ 0 & F \end{pmatrix}$ where D is diagonalizable over K . One has $ND - DN = N^2 - N^3$. According to [1, Corollary 1], $ND = DN$ and $N^2(I - N) = 0$, that implies $N^2 = 0$ and we are done.
- According to Lemma 2, $(A - (n-1)I) \cdots (A - I)AXu = \phi_{n-1}(X)X^2(I - X)^nu = 0$, that is equivalent to the required result. \square

Lemma 3. *Let $\lambda = 0$, ϕ be a polynomial such that $\phi(1) \neq 0$ and $s, t \geq 0$ be distinct integers. Then there is a polynomial ψ such that $\psi(1) \neq 0$ and*

$$(A - sI)\phi(X)X^2(I - X)^tu = \psi(X)X^2(I - X)^tu.$$

Proof. As in the proof of Lemma 2, we obtain

$$\psi(x) = -\phi'(x)X^2(I - X) - 2\phi(x)X(I - X) + t\phi(x)X^2 - s\phi(x).$$

Thus $\psi(1) = (t - s)\phi(1) \neq 0$. \square

Proposition 3. *If $\lambda + s$ is not an eigenvalue of A , then*

$$Xu \in \bigoplus_{0 \leq i \leq s-1} \ker(A - (i + \lambda)I).$$

Proof. We may assume that $\lambda = 0$. Suppose that $(A - (s-1)I) \cdots (A - I)AXu \neq 0$; according to Lemma 2, this is equivalent to

$$\phi_{s-1}(X)X^2(I - X)^su \neq 0 \text{ with } \phi_{s-1}(1) \neq 0.$$

With respect to the matrix X , the minimal polynomial of u has the form $X^r(I - X)^{k+1}$ where $k \geq s$ and $r \leq 2$. Then $(A - kI) \cdots (A - I)AXu = \phi_k(X)X^2(I - X)^{k+1}u = 0$. Since $A - sI$ is invertible, one has

$$(A - kI) \cdots (A - (s+1)I)\phi_{s-1}(X)X^2(I - X)^su = 0.$$

Since $\phi_{s-1}(1) \neq 0$, using repeatedly Lemma 3, the previous equality can be written

$$\psi(X)X^2(I - X)^su = 0 \text{ where } \psi \text{ is a polynomial such that } \psi(1) \neq 0.$$

Therefore, the minimal polynomial of u divides $X^2(I - X)^s$, that is contradictory. \square

Now, we can deduce **Theorem 1**.

Proof. This follows from Proposition 3 and the construction of the $(B_r)_r$. \square

Remark 1. *To solve Eq (2), it suffices to solve it when A has the form of a matrix U_r with $\lambda_r = 0$.*

Assume that $A \in M_n(K)$ is non-derogatory and has k distinct eigenvalues; according to [1, Theorem 5], the algebraic variety of the *nilpotent* solutions $X \in M_n(K)$ of Eq (5) has dimension $n - k$. We look at the dimension of the set of all solutions of Eq (2) in a particular case.

Lemma 4. *The algebraic variety $\{X \in M_n(K) \mid X^2 = 0\}$ has dimension $\lfloor n^2/2 \rfloor$.*

Proof. Any solution X is similar to a matrix in the form $\text{diag}(U_1, \dots, U_k, 0_{n-2k})$, where $U_i = J_2$. We seek the dimension of the similarity class of X as a function of k . One has $\dim(\text{im}(X)) = k$ and $\text{im}(X) \subset \ker(X)$; firstly, the choice of $\ker(X)$ depends on $k(n-k)$ parameters; secondly, the choice of a subspace of dimension k of $\ker(X)$ depends on $k(n-2k)$ parameters; let F be a complementary of $\ker(X)$; finally the choice of an isomorphism $F \rightarrow \text{im}(X)$ depends on k^2 parameters. Thus the dimension of the similarity class is $2k(n-k)$. The maximum value of the previous dimension is obtained for $k = \lfloor n/2 \rfloor$. \square

Lemma 5. *The algebraic variety $\{X \in M_n(K) \mid X^2 - X^3 = 0\}$ has dimension $\lfloor 2n^2/3 \rfloor$.*

Proof. Any solution is similar to a matrix in the form $\text{diag}(U_1, \dots, U_k, 0_t, I_{n-2k-t})$, where $U_i = J_2$. We seek the dimension of the similarity class of X as a function of k . Firstly, the choice of the generalized eigenspace of the eigenvalue 0 depends on $(n-2k-t)(2k+t)$; secondly, the choice of the eigenspace of the eigenvalue 1 depends on $(n-2k-t)(2k+t)$; finally, the choice of the restriction of X to the generalized eigenspace of the eigenvalue 0 depends on $2k(k+t)$ parameters (cf. the proof of Lemma 4). Thus the dimension of the similarity class is $r_n(k, t) = 2n(t+2k) - 6k^2 - 6kt - 2t^2$. If k, t were real variables, then a free extremum of r_n is reached for $k = n/3, t = 0$ and is $2n^2/3$. On the boundary, $t = n - 2k$ and $r_n = 2k(n-k)$; the maximum of r_n is reached for $k = n/2$ and is $n^2/2$. Finally, since k, t are integers, the maximum of r_n is reached in a neighborhood of $(n/3, 0)$ and is at most $\lfloor 2n^2/3 \rfloor$. In fact the previous value is reached, for example, when $k = \lfloor n/3 \rfloor$ and $t = 1$ if $n = 2 \bmod 3$, $t = 0$ otherwise. \square

Now we prove **Theorem 2**.

Proof. According to Theorem 1, a solution of Eq (2) is in the form $X = \begin{pmatrix} P & Q \\ 0 & S \end{pmatrix}$

where $P \in M_p(K), S \in M_q(K), P^2 - P^3 = 0, S^2 - S^3 = 0$.

- Case 1. $XA = AX$. According to Lemma 5, a couple solution (P, S) depends at most on $\lfloor 2p^2/3 \rfloor + \lfloor 2q^2/3 \rfloor$ parameters that is less than $\rho - 1$.
- Case 2. $XA \neq AX$. According to the proof of Proposition 1, $\phi(Q) = -Q$ where $\phi + I$ is singular, that is, there are $\lambda \in \sigma(P), \mu \in \sigma(S)$ such that $\lambda = \mu$ and $f'(\lambda) = -1$; thus $\sigma(P)$ and $\sigma(S)$ contain each at least once the eigenvalue 1. Since the eigenvalue 1 of P or S has index 1, the eigenvalue -1 of ϕ has also index 1. In particular, when P, S are fixed solutions, the dimension d of the vector space of solutions Q is the number of eigenvalues of ϕ equal to -1 . Finally, $d = \tau_1 \tau_2$ where $\tau_1 = \dim(\ker(P - I))$, $\tau_2 = \dim(\ker(S - I))$. We calculate the maximal dimension $\delta(k_1, k_2, \tau_1, \tau_2)$ of the triple (P, Q, S) , solutions of Eq (2), such that P, S have Jordan normal forms

$$\text{diag}(U_1, \dots, U_{k_1}, 0_{p-2k_1-\tau_1}, I_{\tau_1}), \text{diag}(U_1, \dots, U_{k_2}, 0_{q-2k_2-\tau_2}, I_{\tau_2}).$$

According to the proof of Lemma 5, $r_p(k_1, \tau_1) = 2p(p - \tau_1) - 2(p - \tau_1)^2 + 2k_1(p - \tau_1 - k_1)$ and $r_q(k_2, \tau_2) = 2q(q - \tau_2) - 2(q - \tau_2)^2 + 2k_2(q - \tau_2 - k_2)$. Therefore $\delta(k_1, k_2, \tau_1, \tau_2) = r_p(k_1, \tau_1) + r_q(k_2, \tau_2) + \tau_1 \tau_2$. If k_1, k_2, τ_1, τ_2 were real variable, then a free extremum of δ is reached in the following point of the boundary

$$k_1 = (5p - q)/16 \geq 0, k_2 = (5q - p)/16 \geq 0, \tau_1 = (q + 3p)/8, \tau_2 = (3q + p)/8$$

and is $M_{p,q} = (11p^2 + 11q^2 + 2pq)/16$. The previous value is also the maximal value of δ on the boundary. Since k_1, k_2, τ_1, τ_2 are integers, the required dimension ν is at most $\lfloor M_{p,q} \rfloor = \rho$. More precisely $\nu = \rho$ except in the following cases ($\approx 28\%$ of

the couples (p, q) where $\nu = \rho - 1$.

Below, the values of p, q such that $\rho = \nu - 1$ are given mod 16 and $\epsilon = \pm 1$.

$$\begin{aligned} p &= 0, \quad q = 4, 5, 8, 11, 12, \\ \epsilon p &= 1, \quad \epsilon q = 11, 12, 14, 15, \\ \epsilon p &= 2, \quad \epsilon q = 2, 5, 6, 14, 15, \\ \epsilon p &= 3, \quad \epsilon q = 5, 6, 8, 9, \\ \epsilon p &= 4, \quad \epsilon q = 0, 8, 9, 12, 15, \\ \epsilon p &= 5, \quad \epsilon q = 0, 2, 3, 15, \\ \epsilon p &= 6, \quad \epsilon q = 2, 3, 6, 9, 10, \\ \epsilon p &= 7, \quad \epsilon q = 9, 10, 12, 13, \\ p &= 8, \quad q = 0, 3, 4, 12, 13. \end{aligned}$$

□

Remark 2. *i) Note that, if $p = q$, then $\rho = \lfloor 3p^2/2 \rfloor$. According to the previous table, the required dimension ν is ρ , except if $p = \pm 2$ or $\pm 6 \pmod{16}$, where $\nu = \rho - 1$.*

ii) Clearly, if the inequality $\frac{1}{5} \leq \frac{p}{q} \leq 5$ is not satisfied, then ν can be far from ρ . For instance, if $p = 2, q = 26$, then $\nu = \rho - 6$.

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